

Simple characterization of positive linear maps preserving continuity of the von Neumann entropy and beyond

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Abstract

We show that a positive linear map preserves local continuity (convergence) of the entropy if and only if it preserves finiteness of the entropy, i.e. transforms operators with finite entropy to operators with finite entropy. The last property is equivalent to the boundedness of the output entropy of a map on the set of pure states.

This criterion is used for analysis of tensor products of quantum channels preserving continuity of the entropy.

Some applications to the entanglement theory are considered.

1 Preliminaries

Linear completely positive maps between Banach spaces of trace-class operators play a basic role in description of evolution of quantum systems and of measurements of their parameters. Such maps are called *quantum channels* (respectively, *quantum operations*) if they preserve (respectively, don't increase) the trace of any input positive operator [6].

In analysis of information properties of infinite-dimensional quantum channels and operations the question about continuity of their output entropy as a function of input states naturally appears [15]. In this note we give a characterization of positive maps (in particular, quantum channels and operations) for which continuity of the entropy on a set of input operators (states) implies continuity of the output entropy on this set.

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Let \mathcal{H} be a separable Hilbert space, $\mathfrak{T}(\mathcal{H})$ the Banach space of trace class operators in \mathcal{H} , $\mathfrak{T}_+(\mathcal{H})$ the cone of positive operators in $\mathfrak{T}(\mathcal{H})$ and $\mathfrak{S}(\mathcal{H})$ the set of *quantum states* – operators in $\mathfrak{T}_+(\mathcal{H})$ with unit trace [6].

The *von Neumann entropy* of a quantum state $\rho \in \mathfrak{S}(\mathcal{H})$ is defined by the formula $H(\rho) = \text{Tr} \eta(\rho)$, where $\eta(x) = -x \log x$ for $x > 0$ and $\eta(0) = 0$. It is a concave lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ taking values in $[0, +\infty]$ (see [6, 8, 17]). The homogeneous extension of the von Neumann entropy to the cone $\mathfrak{T}_+(\mathcal{H})$ is given by the formula:

$$H(\rho) \doteq [\text{Tr} \rho] H\left(\frac{\rho}{\text{Tr} \rho}\right) = \text{Tr} \eta(\rho) - \eta(\text{Tr} \rho).$$

By using theorem 11.10 in [10] and a simple approximation it is easy to obtain the following inequality

$$H\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k H(\rho_k) + S(\{p_k\}_k), \quad (1)$$

valid for any finite or countable collection $\{\rho_k\}$ of positive operators in the unit ball of $\mathfrak{T}(\mathcal{H})$ and any probability distribution $\{p_k\}$, where

$$S(\{p_k\}) \doteq \sum_k \eta(p_k) - \eta\left(\sum_k p_k\right) \quad (2)$$

is the homogeneous extension of the Shannon entropy to the positive cone of ℓ_1 . Inequality (1) implies the following one

$$H\left(\sum_k \rho_k\right) \leq \sum_k H(\rho_k) + S(\{\text{Tr} \rho_k\}_k), \quad (3)$$

valid for any finite or countable collection $\{\rho_k\}$ of positive operators in $\mathfrak{T}(\mathcal{H})$ with finite $\sum_k \text{Tr} \rho_k$. If $\text{supp} \rho_k \perp \text{supp} \rho_j$ for all $k \neq j$ then " = " holds in (3).¹

The *quantum relative entropy* for positive trace class operators ρ and σ can be defined as

$$H(\rho \| \sigma) = \sum_i \langle i | \rho \log \rho - \rho \log \sigma + \sigma - \rho | i \rangle, \quad (4)$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of the state ρ and it is assumed that $H(\rho \| \sigma) = +\infty$ if $\text{supp} \rho$ is not contained in $\text{supp} \sigma$ [8].

¹The support $\text{supp} \rho$ of a positive operator ρ is the orthogonal complement to its kernel.

2 PCE-property and its characterization

2.1 General case

For an arbitrary positive linear map $\Phi: \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ the output entropy $H_\Phi(\rho) \doteq H(\Phi(\rho))$ is a concave lower semicontinuous function on the cone $\mathfrak{T}_+(\mathcal{H}_A)$ taking values in $[0, +\infty]$.

The class of positive maps characterised by continuity of the output entropy $H_\Phi(\rho)$ on any subset of the cone $\mathfrak{T}_+(\mathcal{H}_A)$ on which the input entropy $H(\rho)$ is continuous was considered in [15]. This property can be called *preserving continuity of the entropy (PCE)* under action of a map Φ . A simple characterization of the PCE-property is given by the following theorem (which is a strengthened version of Theorem 2 in [15]).

Theorem 1. *Let Φ be a positive linear map from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$. The following properties are equivalent:*

(i) Φ preserves continuity of the entropy, i.e.

$$\lim_{n \rightarrow \infty} H(\rho_n) = H(\rho_0) < +\infty \quad \Rightarrow \quad \lim_{n \rightarrow \infty} H_\Phi(\rho_n) = H_\Phi(\rho_0) < +\infty$$

for any sequence $\{\rho_n\} \subset \mathfrak{T}_+(\mathcal{H}_A)$ converging to an operator ρ_0 ;

(ii) Φ preserves finiteness of the entropy, i.e.

$$H(\rho) < +\infty \quad \Rightarrow \quad H_\Phi(\rho) < +\infty$$

for any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$;

(iii) the function $H_\Phi(\rho)$ is bounded on the set $\text{ext}\mathfrak{S}(\mathcal{H}_A)$ of pure states.

Proof. We may assume that Φ does not increase a trace of any positive operator.

It is clear that (i) implies (ii).

By using inequality (1) and the spectral decomposition $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$ we obtain

$$H_\Phi(\rho) \leq \sum_i p_i H_\Phi(|\varphi_i\rangle\langle\varphi_i|) + H(\rho) \leq C \|\rho\|_1 + H(\rho), \quad (5)$$

where $C \doteq \sup_{\rho \in \text{ext}\mathfrak{S}(\mathcal{H}_A)} H_\Phi(\rho)$. This shows that (iii) implies (ii).

To prove the implication (ii) \Rightarrow (iii) assume that for each natural n there exists a pure state ρ_n such that $H_\Phi(\rho_n) \geq 2^n$. It follows from (1) that the

state $\rho_0 = \sum_{n \in \mathbb{N}} 2^{-n} \rho_n$ has finite entropy, while the concavity of the function H_Φ implies $H_\Phi(\rho_0) \geq \sum_{n \in \mathbb{N}} 2^{-n} H_\Phi(\rho_n) = +\infty$.

To prove the nontrivial implication (iii) \Rightarrow (i) it suffices to show that (iii) implies continuity of the function H_Φ on the set $\text{ext}\mathfrak{S}(\mathcal{H}_A)$ and to apply Theorem 2 in [15]. But we will give an independent proof of this implication based on the recently established monotonicity of the relative entropy under positive linear maps [9].

Show first that (iii) implies continuity of the function H_Φ on the set

$$\mathfrak{S}_k(\mathcal{H}_A) = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \text{rank } \rho \leq k\} \quad (6)$$

for any k by using Winter's modification of the Alicki-Fannes method [1, 19].

Let ρ and σ be different states in $\mathfrak{S}_k(\mathcal{H}_A)$ and $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$. Following [19] introduce the state $\omega^* = (1 + \varepsilon)^{-1}(\rho + [\sigma - \rho]_+)$. Then

$$\frac{1}{1 + \varepsilon} \rho + \frac{\varepsilon}{1 + \varepsilon} \tau_- = \omega^* = \frac{1}{1 + \varepsilon} \sigma + \frac{\varepsilon}{1 + \varepsilon} \tau_+,$$

where $\tau_+ = \varepsilon^{-1}[\rho - \sigma]_+$ and $\tau_- = \varepsilon^{-1}[\rho - \sigma]_-$ are states in $\mathfrak{S}(\mathcal{H}_A)$. Since ρ and σ are different states in $\mathfrak{S}_k(\mathcal{H}_A)$, the rank of the states τ_+ and τ_- does not exceed $2k - 1$. It follows from (5) that

$$H_\Phi(\tau_\pm) \leq C + \log(2k - 1). \quad (7)$$

The concavity of the entropy and inequality (1) imply

$$pH_\Phi(\varrho) + (1 - p)H_\Phi(\varsigma) \leq H_\Phi(p\varrho + (1 - p)\varsigma) \leq pH_\Phi(\varrho) + (1 - p)H_\Phi(\varsigma) + h_2(p)$$

for any $p \in (0, 1)$ and any states ϱ and ς , where $h_2(p) = \eta(p) + \eta(1 - p)$.

By applying this double inequality to the above convex decompositions of ω_* we obtain

$$(1 - p)[H_\Phi(\rho) - H_\Phi(\sigma)] \leq p[H_\Phi(\tau_+) - H_\Phi(\tau_-)] + h_2(p)$$

and

$$(1 - p)[H_\Phi(\sigma) - H_\Phi(\rho)] \leq p[H_\Phi(\tau_-) - H_\Phi(\tau_+)] + h_2(p),$$

where $p = \frac{\varepsilon}{1 + \varepsilon}$.

These inequalities and upper bound (7) show that

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \varepsilon(C + \log(2k - 1)) + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right).$$

This implies (uniform) continuity of the function $H_\Phi(\rho)$ on the set $\mathfrak{S}_k(\mathcal{H}_A)$.

We will prove (i) by using the approximation technique proposed in [14]. By Proposition 3 in [14] any concave lower semicontinuous nonnegative function f on $\mathfrak{S}(\mathcal{H})$ is a pointwise limit of the nondecreasing sequence $\{\hat{f}_k\}$ of concave lower semicontinuous nonnegative functions on $\mathfrak{S}(\mathcal{H})$ defined as follows

$$\hat{f}_k(\rho) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i f(\rho_i), \quad \rho \in \mathfrak{S}(\mathcal{H}), \quad (8)$$

where $\mathcal{P}_k(\rho)$ is the set of all countable ensembles of states in $\mathfrak{S}_k(\mathcal{H})$ with the average state ρ . If the function f is continuous and bounded on the set $\mathfrak{S}_k(\mathcal{H})$ for any natural k then all the functions \hat{f}_k are continuous on $\mathfrak{S}(\mathcal{H})$.² So, in this case Dini's lemma implies the following criterion for local continuity of the function f : *the function f is continuous on a compact subset of $\mathfrak{S}(\mathcal{H})$ if and only if the sequence $\{\hat{f}_k\}$ uniformly converges to the function f on this subset.*

This criterion gives a powerful method for analysis of continuity of the von Neumann entropy (described in [14]), since in the case $f = H$ the difference $\Delta_k^H(\rho) = H(\rho) - \hat{H}_k(\rho)$ between the von Neumann entropy and its k -approximator \hat{H}_k is expressed via the quantum relative entropy as follows

$$\Delta_k^H(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\rho_i \| \rho). \quad (9)$$

Since we have shown that (iii) implies continuity and boundedness of the function H_Φ on the set $\mathfrak{S}_k(\mathcal{H}_A)$ for each k , we can use the above criterion of local continuity in the case $f = H_\Phi$. The concavity of the function $\eta(x) = -x \log x$ implies that

$$\Delta_k^{H_\Phi}(\rho) \doteq H_\Phi(\rho) - [\widehat{H_\Phi}]_k(\rho) \leq \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)). \quad (10)$$

By the monotonicity of the relative entropy (defined by formula (4)) under trace-non-increasing positive linear maps (see the Appendix) it follows from (9) and (10) that

$$\Delta_k^{H_\Phi}(\rho) \leq \Delta_k^H(\rho) \quad \text{for all } \rho \in \mathfrak{S}(\mathcal{H}_A).$$

²This is a corollary of the strong stability of the set $\mathfrak{S}(\mathcal{H})$ [14].

Hence, uniform convergence of the sequence $\{\hat{H}_k\}$ to the function H on any subset of $\mathfrak{S}(\mathcal{H}_A)$ implies uniform convergence of the sequence $\{[\widehat{H_\Phi}]_k\}$ to the function H_Φ on this subset. By combining this observation with the above criterion of local continuity in the cases $f = H$ and $f = H_\Phi$ we obtain the implication in (i) for any sequence $\{\rho_n\}$ of states converging to a state ρ_0 . Since the functions H and H_Φ are homogeneous (of degree 1) to prove the general form of (i) it suffices to show that $\lim_n H_\Phi(\rho_n) = 0$ for any sequence $\{\rho_n\}$ in $\mathfrak{T}_+(\mathcal{H}_A)$ converging to the zero operator such that $\lim_n H(\rho_n) = 0$. This can be done by using (5). \square

The arguments from the proof of Theorem 1 imply the following

Corollary 1. *If $H_\Phi(\rho) \leq C < +\infty$ for any pure state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ then the function H_Φ is uniformly continuous on the set $\mathfrak{S}_k(\mathcal{H}_A)$ defined in (6) for any natural k and*

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \varepsilon(C + \log(\text{rank}\rho + \text{rank}\sigma - 1)) + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)$$

for any finite rank states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$, where $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$.

The simplest PCE-maps are maps with finite-dimensional output and unitary transformations, i.e. maps of the form $\Phi(\rho) = U\rho U^*$, where U is an isometry from \mathcal{H}_A into \mathcal{H}_B . More interesting examples are considered in the next subsections.

2.2 Completely positive maps

A linear map $\Phi : \mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$ is called completely positive (CP map) if the map $\Phi \otimes \text{Id}_\mathcal{K}$ is positive for any Hilbert space \mathcal{K} . CP maps play important role in quantum theory. Trace preserving CP maps called *quantum channels* describe evolution of a state of an open quantum system, trace non-increasing CP maps called *quantum operations* are also used in quantum theory, in particular, for mathematical description of quantum measurements [6, 18].

In this subsection we apply the PCE-criterion in Theorem 1 to the class of quantum channels and operations. It is well known that any quantum operation (correspondingly, channel) Φ has the Kraus representation

$$\Phi(\rho) = \sum_k V_k \rho V_k^*, \quad (11)$$

where $\{V_k\}$ is a collection of linear operators from \mathcal{H}_A to \mathcal{H}_B such that $\sum_k V_k^* V_k \leq I_{\mathcal{H}_A}$ (correspondingly, $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$). The minimal number of nonzero summands in representation (11) is called *Choi rank* of the operation Φ [6, 10, 18].

If a quantum operation Φ has finite Choi rank m then it follows from (3) that $H_\Phi(\rho) \leq \log m$ for any pure state ρ . Hence, Theorem 1 shows that Φ is a PCE-operation, while Corollary 1 gives the continuity bound

$$|H_\Phi(\rho) - H_\Phi(\sigma)| \leq \varepsilon \log[m(\text{rank}\rho + \text{rank}\sigma - 1)] + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)$$

valid for any finite rank states ρ and σ , where $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$.

Theorem 1 implies the following conditions of the PCE-property for quantum operations with infinite Choi rank.

Corollary 2. *A quantum operation Φ having representation (11) possesses the PCE-property if one of the following conditions holds:*

- a) *the function $\varphi \mapsto S(\{\|V_k|\varphi\rangle\|^2\}_k)$ is bounded on the unit ball of \mathcal{H}_A ;³*
- b) *$\sum_k \|V_k\|^2$ and $S(\{\|V_k\|^2\}_k)$ are finite;*
- c) *there exists a sequence $\{h_k\}$ of nonnegative numbers such that*

$$\left\| \sum_k h_k V_k^* V_k \right\| < +\infty \quad \text{and} \quad \sum_k e^{-h_k} < +\infty.$$

If $\text{Ran}V_k \perp \text{Ran}V_j$ for all $k \neq j$ then a) is a necessary and sufficient condition of the PCE-property of the operation Φ .

Proof. It follows from (3) that condition a) implies boundedness of the function H_Φ on the set $\text{ext}\mathfrak{S}(\mathcal{H}_A)$. The necessity of this condition in the case $\text{Ran}V_k \perp \text{Ran}V_j$ follows from the remark after inequality (3).

Conditions b) and c) are easily-verified sufficient conditions for a). The implication b) \Rightarrow a) is obvious. To prove the implication c) \Rightarrow a) it suffices to note that the extended Shannon entropy is bounded on the subset of the cone $[\ell_1]_+$ consisting of vectors $\{p_k\}$ such that $\sum_k h_k p_k \leq C$ for any C . \square

Remark 1. Condition b) in Corollary 2 is the most easily verified but is too rough because it depends only on the norms of the Kraus operators.

³ S is the homogeneous extension of the Shannon entropy to the positive cone of ℓ_1 defined in (2).

Condition c) is more subtle, since it takes "geometry" of the sequence $\{V_k\}$ into account. This is confirmed by the following example.

Example 1. Let $\{P_k\}_{k>1}$ be any sequence of mutually orthogonal projectors in $\mathfrak{B}(\mathcal{H}_A)$ and $\alpha \in [0, \log 2]$. Consider the quantum channel

$$\Phi_\alpha(\rho) = \sum_{k \geq 1} c_k P_k \rho P_k,$$

where $c_1 = 1$, $c_k = \alpha / \log k$ for $k > 1$ and $P_1 = \sqrt{I_{\mathcal{H}_A} - \sum_{k>1} c_k P_k}$. Condition c) in Corollary 2 shows that Φ_α is a PCE-channel, while condition b) is not valid in this case.

Sometimes it is possible to prove the PCE-property of a quantum channel (operation) without using its Kraus representation.

Example 2. Let \mathcal{H}_a be the Hilbert space $\mathcal{L}_2([-a, +a])$, where $a < +\infty$, and $\{U_t\}_{t \in \mathbb{R}}$ be the group of unitary operators in \mathcal{H}_a defined as follows

$$(U_t \varphi)(x) = e^{-itx} \varphi(x), \quad \forall \varphi \in \mathcal{H}_a.$$

For given probability density function $p(t)$ consider the quantum channel

$$\Phi_p^a : \mathfrak{T}(\mathcal{H}_a) \ni \rho \mapsto \int_{-\infty}^{+\infty} U_t \rho U_t^* p(t) dt \in \mathfrak{T}(\mathcal{H}_a).$$

In [12] it is shown that the function $H_{\Phi_p^a}$ is bounded and continuous on the set $\text{ext}\mathfrak{S}(\mathcal{H}_a)$ provided that the differential entropy of the distribution $p(t)$ is finite and that the function $p(t)$ is bounded and monotonic on $(-\infty, -b]$ and on $[+b, +\infty)$ for sufficiently large b . So, in this case Φ_p^a is a PCE-channel with infinite Choi rank.

Note that Theorem 1 makes it possible to essentially simplify the proof in [12], since it shows that boundedness of the function $H_{\Phi_p^a}$ on the set $\text{ext}\mathfrak{S}(\mathcal{H}_a)$ implies its continuity on this set.

2.3 Types of PCE-channels and tensor products

It follows from the definition that the class of PCE-channels is closed under composition: if $\Phi : A \rightarrow B$ and $\Psi : B \rightarrow C$ are PCE-channels then $\Psi \circ \Phi$ is a PCE-channel. By using Theorem 1 it is easy to show that any convex mixture of PCE-channels between given quantum systems is a PCE-channel. But the class of PCE-channels is not closed under tensor products: the tensor product

of the identity channel and completely depolarising channel with pure output state is not a PCE-channel. Moreover, it is easy to see that the tensor square of the PCE-channel $\rho \mapsto (1-p)\rho + p\sigma$, where σ is a given pure state, is not a PCE-channel. On the other hand, there are nontrivial PCE-channels Φ and Ψ such that $\Phi \otimes \Psi$ is a PCE-channel (see below). Note also that the PCE-property of a channel $\Phi \otimes \Psi$ obviously implies the same property of the channels Φ and Ψ .

To analyse the PCE-property of tensor products we will introduce a classification of PCE-channels based on the notion of a complementary channel.

For any quantum channel $\Phi : A \rightarrow B$ the Stinespring theorem implies existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A).$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \hat{\Phi}(\rho) = \text{Tr}_B V \rho V^* \in \mathfrak{T}(\mathcal{H}_E)$$

is called *complementary* to the channel Φ [6, Ch.6].

Since the functions H_Φ and $H_{\hat{\Phi}}$ coincide on the sets of pure states in $\mathfrak{S}(\mathcal{H}_A)$, Theorem 1 implies the following

Corollary 3. *Φ is a PCE-channel if and only if $\hat{\Phi}$ is a PCE-channel.*

The function $H_{\hat{\Phi}}$ is an important entropic characteristic of a channel Φ called the entropy exchange of Φ [6, 18].

Proposition 1. *Any PCE-channel $\Phi : A \rightarrow B$ belongs to one of the classes \mathcal{A}, \mathcal{B} and \mathcal{C} characterized, respectively, by the conditions:*

- a) $H_\Phi(\rho) < +\infty$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$;
- b) $H_{\hat{\Phi}}(\rho) < +\infty$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$;
- c) $\sup_{\rho \in \text{ext}\mathfrak{S}(\mathcal{H}_A)} H_\Phi(\rho) < +\infty$, but $H_\Phi(\rho) = H_{\hat{\Phi}}(\sigma) = +\infty$ for some $\rho, \sigma \in \mathfrak{S}(\mathcal{H}_A)$

The classes \mathcal{A}, \mathcal{B} and \mathcal{C} are convex. Convex mixture of channels from different classes belongs to the class \mathcal{C} .

The classes \mathcal{A} and \mathcal{B} are closed under compositions⁴, while the class \mathcal{C} is not.

⁴in the sense described at the begin of this subsection

Proof. By Theorem 1 in [15] condition a) (corresp., b)) implies boundedness and continuity of the function H_Φ (corresp., $H_{\hat{\Phi}}$) on the set $\mathfrak{S}(\mathcal{H}_A)$. So, any of these conditions implies the PCE-property of Φ . The inequality

$$H(\rho) \leq H_\Phi(\rho) + H_{\hat{\Phi}}(\rho), \quad \rho \in \mathfrak{S}(\mathcal{H}_A)$$

(which follows from subadditivity of the entropy) shows that the classes \mathcal{A} and \mathfrak{B} are disjoint. The convexity of all the classes can be established by using basic properties of the entropy and the relation

$$H_{\hat{\Phi}}(\rho) = H_{\Phi \otimes \text{Id}_R}(\hat{\rho}), \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad (12)$$

where $\hat{\rho}$ is a purification of ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ [6].

The closedness of the class \mathcal{A} under compositions is obvious. To prove the same property of the class \mathfrak{B} we will use Proposition 2 below (proved independently).

Assume that $\Phi : A \rightarrow B$ and $\Psi : B \rightarrow C$ are PCE-channels of the class \mathfrak{B} . Let R be an infinite-dimensional quantum system. By Proposition 2A $\Phi \otimes \text{Id}_R$ and $\Psi \otimes \text{Id}_R$ are PCE-channels. So, $\Psi \otimes \text{Id}_R \circ \Phi \otimes \text{Id}_R = (\Psi \circ \Phi) \otimes \text{Id}_R$ is a PCE-channel. By Proposition 2B $\Psi \circ \Phi$ is a PCE-channel from class \mathfrak{B} .

To show that the class \mathfrak{C} is not closed under compositions consider the PCE-channels

$$\Phi(\rho) = P\rho P \oplus [\text{Tr}\bar{P}\rho]\sigma \quad \text{and} \quad \Psi(\rho) = [\text{Tr}P\rho]\varsigma \oplus \bar{P}\rho\bar{P},$$

where P and $\bar{P} = I_{\mathcal{H}} - P$ are infinite rank projectors, σ and ς are pure states such that $\bar{P}\sigma\bar{P} = \sigma$ and $P\varsigma P = \varsigma$. These channels belong to the class \mathfrak{C} . The simplest way to show this is to note that $\Phi^{\otimes 2}$ and $\Psi^{\otimes 2}$ are not PCE-channels and to use Proposition 2 below. It is easy to see that $\Psi \circ \Phi$ is a completely depolarizing channel belonging to the class \mathcal{A} . \square

The class \mathcal{A} consists of channels with continuous output entropy. The simplest example of such channels is the completely depolarizing channel $\rho \mapsto [\text{Tr}\rho]\sigma$, where σ is a given state with finite entropy. More interesting channels from the class \mathcal{A} are considered in [15, Sect.3].

The class \mathfrak{B} contains the identity channel and all channels with finite Choi rank. Channel from the class \mathfrak{B} having infinite Choi rank is presented in the above Example 2. The finiteness of the function $H_{\hat{\Phi}_p^a}$ in this case can be shown by using the explicit expression for the channel $\hat{\Phi}_p^a$ presented at the end of Section 4 in [12] and the results in [15, Sect.3].

The simplest example of a PCE-channel from the class \mathfrak{C} is the channel $\rho \mapsto (1-p)\rho + p\sigma$, where σ is a pure state, – the convex mixture of the identity channel (from the class \mathfrak{B}) and the completely depolarising channel (from the class \mathfrak{A}).

Proposition 2. *Let Φ and Ψ be PCE-channels.*

A) *If both channels Φ and Ψ belong to one of the classes \mathfrak{A} and \mathfrak{B} then $\Phi \otimes \Psi$ is a PCE-channel of the same class.*

B) *If the channels Φ and Ψ belong to different classes then $\Phi \otimes \Psi$ is not a PCE-channel.*

Proof. A) If Φ and Ψ are channels of the class \mathfrak{A} (corresp., \mathfrak{B}) then the functions H_Φ and H_Ψ (corresp., $H_{\widehat{\Phi}}$ and $H_{\widehat{\Psi}}$) are bounded. By subadditivity of the entropy this implies boundedness of the function $H_{\Phi \otimes \Psi}$ (corresp., $H_{\widehat{\Phi \otimes \Psi}} = H_{\widehat{\Phi \otimes \Psi}}$).

B) Let $\Phi : A \rightarrow B$ be a channel of the class \mathfrak{A} and $\Psi : C \rightarrow D$ a channel of one of the classes \mathfrak{B} and \mathfrak{C} . Let ρ be a state in $\mathfrak{S}(\mathcal{H}_C)$ such that $H_\Psi(\rho) = +\infty$ and $\hat{\rho}$ a purification of ρ in $\mathfrak{S}(\mathcal{H}_{AC})$. Since $H_\Phi(\hat{\rho}_A) < +\infty$, the triangle inequality

$$H_{\Phi \otimes \Psi}(\hat{\rho}) \geq |H_\Phi(\hat{\rho}_A) - H_\Psi(\hat{\rho}_C)|$$

shows that $H_{\Phi \otimes \Psi}(\hat{\rho}) = +\infty$.

If channels Φ and Ψ belong, respectively, to the classes \mathfrak{B} and \mathfrak{C} then the similar arguments with the complementary channels $\widehat{\Phi}$ and $\widehat{\Psi}$ show that $\widehat{\Phi \otimes \Psi}$ is not a PCE-channel. This means, by Corollary 3, that $\Phi \otimes \Psi$ is not a PCE-channel. \square

Remark 2. Proposition 2 says nothing about tensor product of channels from the class \mathfrak{C} . There is a conjecture that $\Phi \otimes \Psi$ is not a PCE-channel if one of the channels, say Φ , belongs to the class \mathfrak{C} . If Φ is a convex mixture of channels from the classes \mathfrak{A} and \mathfrak{B} then this assertion directly follows from Proposition 2B, but it is not clear how to prove it in general case.

Corollary 4. *If Φ is a PCE-channel from one of the classes \mathfrak{A} and \mathfrak{B} then $\Phi^{\otimes n}$ is a PCE-channel of the same class for any n .*

2.4 The convex closure of the output entropy

In analysis of informational properties of a quantum channel Φ the convex closure $\overline{\text{co}}H_\Phi$ of its output entropy plays important role [6, 12]. The function

$\overline{\text{co}}H_\Phi$ is defined as the maximal closed (lower semicontinuous) convex function on $\mathfrak{S}(\mathcal{H}_A)$ majorized by the function H_Φ . In finite dimensions $\overline{\text{co}}H_\Phi$ coincides with the convex hull $\text{co}H_\Phi$ of H_Φ – the maximal convex function on $\mathfrak{S}(\mathcal{H}_A)$ majorized by the function H_Φ which is given by the formula

$$\text{co}H_\Phi(\rho) = \inf_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H_\Phi(\rho_i), \quad (13)$$

where the infimum is over all finite ensembles $\{\pi_i, \rho_i\}$ of input states with the average state ρ .

In infinite dimensions the function $\overline{\text{co}}H_\Phi$ coincides with $\text{co}H_\Phi$ only for positive maps (channels) with finite output entropy, but one can assume that it coincides with the σ -convex hull $\sigma\text{-co}H_\Phi$ of H_Φ defined by formula (13) in which the infimum is over all countable ensembles $\{\pi_i, \rho_i\}$ of input states with the average state ρ .

On the other hand, the compactness criterion for families of probability measures on $\mathfrak{S}(\mathcal{H})$ makes it possible to show that

$$\overline{\text{co}}H_\Phi(\rho) = \inf_{\bar{\rho}(\mu)=\rho} \int H_\Phi(\rho) \mu(d\rho), \quad (14)$$

where the infimum is over all Borel probability measures on the set $\mathfrak{S}(\mathcal{H}_A)$ with the barycenter ρ [12, 13]. So, to prove the conjecture $\sigma\text{-co}H_\Phi = \overline{\text{co}}H_\Phi$ it suffices to show that the infimum in (14) can be taken only over all discrete probability measures. We don't know how to prove (or disprove) this conjecture in general⁵, but it seems reasonable to mention that it holds for PCE-channels.

Corollary 5. *If properties (i)-(iii) in Theorem 1 hold for a map Φ then*

A) $\sigma\text{-co}H_\Phi(\rho) = \overline{\text{co}}H_\Phi(\rho)$ for any state $\rho \in \mathfrak{S}(\mathcal{H}_A)$;

B) *the function $\sigma\text{-co}H_\Phi = \overline{\text{co}}H_\Phi$ is continuous and bounded on $\mathfrak{S}(\mathcal{H}_A)$.*

Proof. Since the assumption implies continuity and boundedness of the function H_Φ on the set $\text{ext}\mathfrak{S}(\mathcal{H}_A)$, both assertions of the corollary follow from Proposition 2 in [12]. They can be also derived from Corollary 2 in [13] by using Remark 3 below.

Remark 3. By using the concavity of the function H_Φ one can show that $\sigma\text{-co}H_\Phi = \check{H}_\Phi^\sigma$ and $\overline{\text{co}}H_\Phi = \check{H}_\Phi^\mu$, where \check{H}_Φ^σ and \check{H}_Φ^μ are discrete and

⁵It is shown in [13] that $\sigma\text{-co}f \neq \overline{\text{co}}f$ for a particular lower semicontinuous concave nonnegative unitarily invariant function f on $\mathfrak{S}(\mathcal{H})$, so the above conjecture can not be proved by using only general entropy-type properties of the function H_Φ .

continuous convex roof extensions⁶ of the function $H_\Phi|_{\text{ext}\mathfrak{S}(\mathcal{H}_A)}$ defined, respectively, by the right hand sides of (13) and (14) in which the infima are over all ensembles (measures) consisting of (supported by) pure states [13, Sect.2.3]. Thus, the assertion of Corollary 5 can be reformulated in terms of the functions \check{H}_Φ^σ and \check{H}_Φ^μ (instead of $\sigma\text{-co}H_\Phi$ and $\overline{\text{co}}H_\Phi$).

3 Application to the entanglement theory

Important task of the entanglement theory consists in finding appropriate characteristics of entanglement of composite states and in exploring their properties [4, 7].

If ρ_{AB} is a pure state of a bipartite system AB of any dimension then its entanglement is characterized by the von Neumann entropy of partial states:

$$E(\rho_{AB}) \doteq H(\rho_A) = H(\rho_B).$$

Entanglement of mixed states of a bipartite system AB is characterized by different entanglement measures [5, 11, 16]. One of the most important of them is the Entanglement of Formation (EoF).

In the case of finite-dimensional bipartite system AB the Entanglement of Formation is defined as the convex roof extension to the set $\mathfrak{S}(\mathcal{H}_{AB})$ of the function $\rho_{AB} \mapsto H(\rho_A)$ on the set $\text{ext}\mathfrak{S}(\mathcal{H}_{AB})$ of pure states, i.e.

$$E_F(\rho_{AB}) = \inf_{\sum_i \pi_i \rho_{AB}^i = \rho_{AB}} \sum_i \pi_i H(\rho_A^i), \quad (15)$$

where the infimum is over all ensembles $\{\pi_i, \rho_{AB}^i\}$ of pure states with the average state ρ_{AB} [3].

In infinite dimensions there are two versions E_F^d and E_F^c of the EoF defined, respectively, by using discrete and continuous convex roof extensions, i.e.

$$E_F^d(\rho_{AB}) = \inf_{\sum_i \pi_i \rho_{AB}^i = \rho_{AB}} \sum_i \pi_i H(\rho_A^i), \quad E_F^c(\rho_{AB}) = \inf_{b(\mu) = \rho_{AB}} \int H(\rho_A) \mu(d\rho),$$

where the first infimum is over all countable convex decompositions of the state ρ_{AB} into pure states and the second one is over all Borel probability measures on the set $\text{ext}\mathfrak{S}(\mathcal{H}_{AB})$ with the barycenter ρ_{AB} [13, Sect.5].

⁶The convex roof extension is widely used for construction of different characteristics of states in finite-dimensional quantum systems [7, 11].

The continuous version E_F^c is a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{AB})$ of all states of infinite-dimensional bipartite system possessing basic properties of entanglement measures (including monotonicity under generalized selective measurements) [13]. The discrete version E_F^d seems more preferable from the physical point of view but the assumption $E_F^d \neq E_F^c$ leads to several problems with this version, in particular, it is not clear how to prove its vanishing for countably nondecomposable separable states.⁷

In [13] it is shown that $E_F^d(\rho_{AB}) = E_F^c(\rho_{AB})$ for any state ρ_{AB} such that $\min\{H(\rho_A), H(\rho_B), H(\rho_{AB})\} < +\infty$, but the coincidence of E_F^d and E_F^c on the whole set $\mathfrak{S}(\mathcal{H}_{AB})$ is not proved (as far as I know). It is equivalent to the lower semicontinuity of E_F^d on $\mathfrak{S}(\mathcal{H}_{AB})$ (since E_F^c coincides with the convex closure of the entropy of a partial state).

By applying Corollary 5 and Remark 3 to the channel $\Phi(\rho_{AB}) = \rho_A$ we obtain the following

Proposition 3. *Let \mathcal{K} be a subspace of \mathcal{H}_{AB} such that all unit vectors in \mathcal{K} have bounded entanglement, i.e. $\sup_{\varphi \in \mathcal{K}, \|\varphi\|=1} E(|\varphi\rangle\langle\varphi|) < +\infty$. Then*

- A) $E_F^c(\rho) = E_F^d(\rho)$ for any state ρ in $\mathfrak{S}(\mathcal{K})$;
- B) the function $E_F^c = E_F^d$ is continuous on the set $\mathfrak{S}(\mathcal{K})$.

The existence of nontrivial subspaces satisfying the condition of Proposition 3 follows, by the Stinespring representation, from the existence of PCE-channels of the class \mathfrak{B} with infinite Choi rank and of PCE-channels of the class \mathfrak{C} (see Section 2.3).

Appendix: On monotonicity of the relative entropy under trace-non-increasing positive maps

Recently Muller-Hermes and Reeb established (essentially basing on Beigi's results [2]) the following fundamental property.

Theorem 2. [9] *If Φ is a positive trace-preserving linear map then*

$$H(\Phi(\rho)\|\Phi(\sigma)) \leq H(\rho\|\sigma) \quad \text{for any states } \rho \text{ and } \sigma. \quad (16)$$

⁷In general, the discrete convex roof construction applied to an entropy type function (in the role of H) may give a function which is not equal to zero for countably nondecomposable separable states [13, Rem.6].

Muller-Hermes and Reeb mentioned in [9] that this result is not generalized to trace-non-increasing positive linear maps until we use for all positive trace-class operators the same definition of the relative entropy as for quantum states. But using Lindblad's definition (4) of the relative entropy it is easy to obtain such generalization.

Corollary 6. *If the relative entropy is defined by formula (4) then (16) is valid for any trace-non-increasing positive linear map Φ .*

Proof. Consider the trace-preserving positive map $\Phi'(\rho) = \Phi(\rho) \oplus \Psi(\rho)$ from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B \oplus \mathcal{H}_C)$, where $\Psi(\rho) = [\text{Tr}(\rho - \Phi(\rho))]\tau$ is a positive linear map from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_C)$ determined by a given state τ in $\mathfrak{S}(\mathcal{H}_C)$. Then

$$H(\Phi'(\rho)\|\Phi'(\sigma)) = H(\Phi(\rho)\|\Phi(\sigma)) + H(\Psi(\rho)\|\Psi(\sigma)).$$

The nonnegativity of the relative entropy defined by formula (4) and Theorem 2 imply

$$H(\Phi(\rho)\|\Phi(\sigma)) \leq H(\Phi'(\rho)\|\Phi'(\sigma)) \leq H(\rho\|\sigma)$$

for any states ρ and σ . \square

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References

- [1] R.Alicki, M.Fannes, "Continuity of quantum conditional information", Journal of Physics A: Mathematical and General, V.37, N.5, L55-L57 (2004); arXiv: quant-ph/0312081.
- [2] S.Beigi, "Sandwiched Renyi Divergence Satisfies Data Processing Inequality", Journal of Mathematical Physics 54, 122202 (2013); arXiv:1306.5920.
- [3] C.H.Bennett, D.P.DiVincenzo, J.A.Smolin, W.K.Wootters, "Mixed State Entanglement and Quantum Error Correction", Phys. Rev. A, 54, 3824-3851, (1996).
- [4] J.Eisert, "Entanglement in quantum information theory", arxiv:quant-ph/0610253.

- [5] J.Eisert, Ch.Simon, M.B.Plenio "On the quantification of entanglement in infinite-dimensional quantum systems" J. Phys. A V.35, N.17, 3911-3923 (2002).
- [6] A.S.Holevo "Quantum systems, channels, information. A mathematical introduction", Berlin, DeGruyter, 2012.
- [7] R.Horodecki, P.Horodecki, M.Horodecki, K.Horodecki, "Quantum entanglement", Rev.Mod.Phys. V.81. 865-942 (2009); arXiv:quant-ph/0702225.
- [8] G.Lindblad "Expectation and Entropy Inequalities for Finite Quantum Systems", Comm. Math. Phys. V.39. N.2. 111-119 (1974).
- [9] A.Muller-Hermes, D.Reeb "Monotonicity of the Quantum Relative Entropy Under Positive Maps", arXiv:1512.06117.
- [10] M.A.Nielsen, I.L.Chuang "Quantum Computation and Quantum Information", Cambridge University Press, 2000.
- [11] M.B.Plenio, S.Virmani, "An introduction to entanglement measures", Quantum Inf. Comput., V.7 N.1-2, 1-51 (2007).
- [12] M.E.Shirokov "The Convex Closure of the Output Entropy of Infinite Dimensional Channels and the Additivity Problem", arXiv:quant-ph/0608090.
- [13] M.E.Shirokov "On properties of the space of quantum states and their application to construction of entanglement monotones", Izvestiya: Mathematics, V.74, N.4, 849-882 (2010); arXiv:0804.1515.
- [14] M.E.Shirokov, "Continuity of the von Neumann entropy", Commun. Math. Phys., V.296, N.3, P.625-654 (2010); arXiv:0904.1963.
- [15] M.E.Shirokov, "On continuity of the output entropy of positive maps" Sbornik: Mathematics, 202:10, 1537-1564 (2011); arXiv:1002.0230.
- [16] G.Vidal, "Entanglement monotones", J. Modern Opt. V.47 N.2-3, 355-376 (2000).
- [17] A.Wehrh, "General properties of entropy", Rev. Mod. Phys. 50, 221-250, (1978).

- [18] M.M.Wilde, "From Classical to Quantum Shannon Theory", arXiv:1106.1445 (v.7).
- [19] A.Winter, "Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints", Comm. Math. Phys., V.347, N.1, 291-313 (2016); arXiv:1507.07775 (v.6).